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AN IMPROVED ERROR ESTIMATE FOR REISSNER'S PLATE THEORY

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Abstract—Reissner's theory for the bending of anisotropic, homogeneous plates and plane stress theory are used to construct improved three-dimensional displacement and stress fields. Under specific "regular" boundary conditions on the edge surface, these fields differ from the exact elasticity solutions by terms of the order of the plate thickness cubed.

1. INTRODUCTION

Being two-dimensional as they are plate theories provide only approximations to the solutions of three-dimensional elasticity. An efficient method for evaluating the error involved in a global, mean square sense rests on the hypersphere theorem of Prager and Synge (1947), see Synge (1957). The theorem assumes the knowledge of statically and kinematically admissible solutions and bounds the error in terms of their difference. Nordgren (1971) applied this method to Kirchhoff's classical plate theory finding a relative error proportional to the plate thickness, $O(h)$. Simmonds (1971) and Nordgren (1972) were able to reduce the error to $O(h^2)$ by constructing a displacement field incorporating transverse shear deformation. Corresponding $O(h)$ and $O(h^2)$ error bounds for classical shell theory were obtained by Koiter (1970), Danielson (1971) and Ladevèze (1976).

Nordgren's analysis of Reissner's (1945) plate theory resulted in an $O(h^2)$ error estimate, implying that this higher-order theory offers no essential improvement over the classical theory. Berdichevskii (1973) found that in the absence of surface distributed loads Reissner's theory may bear a smaller error $O(h^3)$. An improved error bound for Reissner's theory as applied to anisotropic plates with high transverse shear deformability was arrived at (Rychter, 1986). In the latter two estimates self-equilibrating surface parallel stresses were found essential, their importance being recognized in a different context by Reissner himself (1975, 1985) and by Rehfield and Valisetty (1984).

This report aims at extending the validity of $O(h^3)$ error estimates in Reissner's theory to general homogeneous, anisotropic plates with midsurface elastic symmetry, carrying arbitrarily distributed face lateral loading. This necessitates in the first place that full Reissner's theory be used with the effect of transverse normal stress explicitly included in the moment constitutive equations, as opposed to Nordgren (1972), Berdichevskii (1973) and Rychter (1986). Secondly, at this level of accuracy stretching of the plate resulting from Poisson's effect cannot be ignored, so that Reissner's sixth-order bending field equations must be considered along with fourth-order equations responsible for stretching. From these two sets of two-dimensional equations improved three-dimensional displacement and stress fields are constructed the relative mean square errors of which with respect to the exact solutions are of $O(h^3)$. This is achieved by taking the in-plane and transverse displacement components as third- and fourth-degree polynomials in the thickness coordinate.

Like earlier works employing the hypersphere theorem the improved error estimate is by necessity restricted to "regular", in Koiter's (1970) terminology, boundary conditions on the cylindrical edge surface. They demand that the edge data should be distributed in conformity with the constructed three-dimensional displacement and stress fields, a requirement hardly ever met in practice. The effect of irregular boundary conditions along the lines of Koiter (1970) is discussed below.

2. FORMULATION OF THE PROBLEM

Plates of constant thickness $2h$ are studied, subjected to distributed transverse loading $p(x_\alpha)$ applied to the upper face $x_3 = h$ where x_α ($\alpha = 1, 2$) and x_3 are in-plane Cartesian coordinates and distance from the middle plane $x_3 = 0$. The true state of stress and displacement in the plate will be approximated by means of statically admissible stresses $\tilde{\boldsymbol{\sigma}}$, kinematically admissible stresses $\hat{\boldsymbol{\sigma}}$ and kinematically admissible displacements $\hat{\mathbf{u}}$.

By definition, the statically admissible stresses $\tilde{\boldsymbol{\sigma}}$ must conform to the traction boundary conditions at the faces

$$\tilde{\sigma}_{\alpha 3}(x_\beta, x_3 = \pm h) = 0 \tag{1}$$

$$\tilde{\sigma}_{33}(x_\beta, x_3 = -h) = 0 \tag{2}$$

$$\tilde{\sigma}_{33}(x_\beta, x_3 = h) = p(x_\beta) \tag{3}$$

and should fulfil the equilibrium equations, with no body forces

$$\tilde{\sigma}_{\alpha\beta,\beta} + \tilde{\sigma}_{\alpha 3,3} = 0 \tag{4}$$

$$\tilde{\sigma}_{\alpha 3,\alpha} + \tilde{\sigma}_{33,3} = 0 \tag{5}$$

the commas denoting partial differentiation with respect to x_α and x_3 and repeated indices implying summation over the range 1, 2.

The kinematically admissible fields $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\sigma}}$ are required to satisfy constitutive equations of the form

$$\hat{\sigma}_{\alpha\beta} = \tfrac{1}{2} D_{\alpha\beta\lambda\eta}(\hat{u}_{\lambda,\eta} + \hat{u}_{\eta,\lambda}) + C_{\alpha\beta}\hat{\sigma}_{33} \tag{6}$$

$$\hat{\sigma}_{\alpha 3} = B_{\alpha 3\beta 3}(\hat{u}_{\beta,3} + \hat{u}_{3,\beta}) \tag{7}$$

$$\hat{\sigma}_{33} = \tfrac{1}{2} B_{33\alpha\beta}(\hat{u}_{\alpha,\beta} + \hat{u}_{\beta,\alpha}) + B_{3333}\hat{u}_{3,3} \tag{8}$$

where

$$D_{\alpha\beta\lambda\eta} = B_{\alpha\beta\lambda\eta} - B_{\alpha\beta 33}B_{33\lambda\eta}/B_{3333} \tag{9}$$

$$C_{\alpha\beta} = B_{\alpha\beta 33}/B_{3333}. \tag{10}$$

Here the material has been taken to be homogeneous, linearly elastic and anisotropic with midplane elastic symmetry. The B are the components of the elasticity tensor, $D_{\alpha\beta\lambda\eta}$ and $C_{\alpha\beta}$ being introduced for notational convenience.

It is well known that by minimizing the distance $\tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}$ between the statically and kinematically admissible solutions one approaches the exact solution. One intends to show that starting from Reissner's theory in conjunction with plane stress theory fields $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ can be found such that $\tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}$ is of $O(h^3)$ relative to $\hat{\boldsymbol{\sigma}}$.

3. TWO-DIMENSIONAL PLATE THEORY

The kinematic variables of Reissner's theory include an average lateral displacement w and rotations b_α, defined as

$$w(x_\beta) = \frac{3}{4h}\int_{-h}^{h} \hat{u}_3(x_\beta, x_3)\,(1 - x_3^2/h^2)\;\mathrm{d}x_3 \tag{11}$$

$$b_\alpha(x_\beta) = \frac{3}{2h^3}\int_{-h}^{h} \hat{u}_\alpha(x_\beta, x_3)x_3\;\mathrm{d}x_3. \tag{12}$$

The corresponding static variables are moments $M_{\alpha\beta}$ and transverse shear forces Q_α

$$M_{\alpha\beta}(x_\lambda) = \int_{-h}^{h} \tilde{\sigma}_{\alpha\beta}(x_\lambda, x_3)x_3\;\mathrm{d}x_3 \tag{13}$$

$$Q_\alpha(x_\lambda) = \int_{-h}^{h} \tilde{\sigma}_{\alpha 3}(x_\lambda, x_3)\;\mathrm{d}x_3. \tag{14}$$

The equilibrium equations in terms of $M_{\alpha\beta}$ and Q_α are

$$M_{\alpha\beta,\beta} = Q_\alpha, \quad Q_{\alpha,\alpha} = -p \tag{15, 16}$$

and the constitutive equations have the form

$$M_{\alpha\beta} = \tfrac{1}{3}h^3 D_{\alpha\beta\lambda\eta}(b_{\lambda,\eta} + b_{\eta,\lambda}) + \tfrac{2}{5}h^2 C_{\alpha\beta}p \tag{17}$$

$$Q_\alpha = \tfrac{5}{3}hB_{\alpha 3\beta 3}(b_\beta + w_{,\beta}). \tag{18}$$

The accompanying boundary conditions prescribe

$$M_{\alpha\beta}n_\beta \text{ or } b_\alpha \quad \text{and} \quad Q_\alpha n_\alpha \text{ or } w \quad \text{on} \quad C \tag{19}$$

where C is the edge of the plate with unit normal n_α.

At the level of accuracy one wishes to achieve it is necessary to introduce variables corresponding to plate stretching in addition to those of bending. These encompass in-plane average displacements v_α, defined as

$$v_\alpha(x_\beta) = \frac{1}{2h}\int_{-h}^{h} \hat{u}_\alpha(x_\beta, x_3)\;\mathrm{d}x_3 \tag{20}$$

and in-plane stress resultants $N_{\alpha\beta}$

$$N_{\alpha\beta}(x_\lambda) = \int_{-h}^{h} \tilde{\sigma}_{\alpha\beta}(x_\lambda, x_3)\;\mathrm{d}x_3. \tag{21}$$

The associated equilibrium equations are

$$N_{\alpha\beta,\beta} = 0 \tag{22}$$

and the constitutive equations read

$$N_{\alpha\beta} = hD_{\alpha\beta\lambda\eta}(v_{\lambda,\eta} + v_{\eta,\lambda}) + hC_{\alpha\beta}p. \tag{23}$$

The boundary conditions to be used in conjunction with these equations require the specification of

$$N_{\alpha\beta}n_\beta \quad \text{or} \quad v_\alpha \quad \text{on} \quad C. \tag{24}$$

The sixth-order bending problem, eqns (15)–(19), and the fourth-order stretching problem, eqns (22)–(24), are not coupled. For compactness they have been recorded as given *a priori* two-dimensional equations but in reality they can be derived by constructing close to each other three-dimensional fields $\tilde{\sigma}$ and $\hat{\sigma}$. The derivation is detailed in the next section.

4. THREE-DIMENSIONAL FIELDS

It will be shown that the requirements are met by the following kinematically admissible displacement field $\hat{\mathbf{u}}$:

$$\hat{u}_\alpha = v_\alpha + zhb_\alpha + (z^2 - \tfrac{1}{3})k_\alpha + (z^3 - \tfrac{3}{5}z)f_\alpha \tag{25}$$

$$\hat{u}_3 = w + zs + \tfrac{1}{10}(5z^2 - 1)g + \tfrac{1}{3}(z^3 - z)l + \tfrac{1}{20}(5z^4 - 6z^2 + \tfrac{27}{35})r$$
$$- (z^4 - 6z^2 - 8z + \tfrac{39}{35})hp/16B_{3333} \tag{26}$$

where $z = x_3/h$. k_α, f_α, s, g, l and r are functions of the surface coordinates x_β given in terms of the basic kinematic variables v_α, w and b_α as

$$s = -\frac{h}{2}C_{\alpha\beta}(v_{\alpha,\beta} + v_{\beta,\alpha}), \quad g = -\frac{h^2}{2}C_{\alpha\beta}(b_{\alpha,\beta} + b_{\beta,\alpha}) \tag{27}$$

$$l = -\frac{h}{2}C_{\alpha\beta}(k_{\alpha,\beta} + k_{\beta,\alpha}), \quad r = -\frac{h}{2}C_{\alpha\beta}(f_{\alpha,\beta} + f_{\beta,\alpha}) \tag{28}$$

$$k_\alpha = -\frac{h}{2}s_{,\alpha}, \quad f_\alpha = -\frac{h}{6}g_{,\alpha} - \frac{5h}{12}(b_\alpha + w_{,\alpha}). \tag{29}$$

Introducing eqns (25) and (26) into the constitutive equations, eqns (6)–(8), and using eqns (18) and (27)–(29) yields the corresponding kinematically admissible stress field $\hat{\sigma}$

$$\hat{\sigma}_{\alpha\beta} = \tfrac{1}{2}D_{\alpha\beta\lambda\eta}[v_{\lambda,\eta} + v_{\eta,\lambda} + zh(b_{\lambda,\eta} + b_{\eta,\lambda}) + (z^2 - \tfrac{1}{3})(k_{\lambda,\eta} + k_{\eta,\lambda})$$
$$+ (z^3 - \tfrac{3}{5}z)(f_{\lambda,\eta} + f_{\eta,\lambda})] + \tfrac{1}{4}(2 + 3z - z^3)C_{\alpha\beta}p \tag{30}$$

$$\hat{\sigma}_{\alpha 3} = \frac{3}{4h}(1 - z^2)Q_\alpha + B_{\alpha 3\lambda 3}[\tfrac{1}{20}(5z^4 - 6z^2 + \tfrac{27}{35})r_{,\lambda} + \tfrac{1}{3}(z^3 - z)l_{,\lambda}]$$
$$- (z^4 - 6z^2 - 8z + \tfrac{39}{35})(B_{\alpha 3\lambda 3}/16B_{3333})hp_{,\lambda} \tag{31}$$

$$\hat{\sigma}_{33} = \tfrac{1}{4}(2 + 3z - z^3)p. \tag{32}$$

The statically admissible stress field $\tilde{\sigma}$ needed is

$$\tilde{\sigma}_{\alpha\beta} = \frac{1}{2h}N_{\alpha\beta} + \frac{3z}{2h^2}M_{\alpha\beta} + \tfrac{1}{20}(3z - 5z^3)C_{\alpha\beta}p$$
$$+ \tfrac{1}{2}D_{\alpha\beta\lambda\eta}[(z^2 - \tfrac{1}{3})(k_{\lambda,\eta} + k_{\eta,\lambda}) + (z^3 - \tfrac{3}{5}z)(f_{\lambda,\eta} + f_{\eta,\lambda})] \tag{33}$$

$$\tilde{\sigma}_{\alpha 3} = \frac{3}{4h}(1-z^2)Q_\alpha - \frac{h}{6}(z^3-z)D_{\alpha\beta\lambda\eta}(k_{\lambda,\eta\beta}+k_{\eta,\lambda\beta})$$

$$- \frac{h}{40}(5z^4-6z^2+1)\,[D_{\alpha\beta\lambda\eta}(f_{\lambda,\eta\beta}+f_{\eta,\lambda\beta}) - \tfrac{1}{2}C_{\alpha\beta}p_{,\beta}] \quad (34)$$

$$\tilde{\sigma}_{33} = \tfrac{1}{4}(2+3z-z^3)p + \frac{h^2}{24}(z^4-2z^2+1)D_{\alpha\beta\lambda\eta}(k_{\lambda,\eta\beta\alpha}+k_{\eta,\lambda\beta\alpha})$$

$$+ \frac{h^2}{40}(z^5-2z^3+z)\,[D_{\alpha\beta\lambda\eta}(f_{\lambda,\eta\beta\alpha}+f_{\eta,\lambda\beta\alpha}) - \tfrac{1}{2}C_{\alpha\beta}p_{,\beta\alpha}]. \quad (35)$$

From eqns (30)–(35) with eqns (17) and (23) the difference $\tilde{\sigma}-\hat{\sigma}$ is found to have the components

$$\tilde{\sigma}_{\alpha\beta} - \hat{\sigma}_{\alpha\beta} = 0 \quad (36)$$

$$\tilde{\sigma}_{\alpha 3} - \hat{\sigma}_{\alpha 3} = -\frac{h}{6}(z^3-z)D_{\alpha\beta\lambda\eta}(k_{\lambda,\eta\beta}+k_{\eta,\lambda\beta}) - \tfrac{1}{3}(z^3-z)B_{\alpha 3\lambda 3}l_{,\lambda}$$

$$- \frac{h}{40}(5z^4-6z^2+1)\,[D_{\alpha\beta\lambda\eta}(f_{\lambda,\eta\beta}+f_{\eta,\lambda\beta}) - \tfrac{1}{2}C_{\alpha\beta}p_{,\beta}]$$

$$- \tfrac{1}{20}(5z^4-6z^2+\tfrac{37}{35})B_{\alpha 3\lambda 3}r_{,\lambda} + (z^4-6z^2-8z+\tfrac{39}{35})\,(B_{\alpha 3\lambda 3}/16B_{3333})hp_{,\lambda} \quad (37)$$

$$\tilde{\sigma}_{33} - \hat{\sigma}_{33} = \frac{h^2}{24}(z^4-2z^2+1)D_{\alpha\beta\lambda\eta}(k_{\lambda,\eta\beta\alpha}+k_{\eta,\lambda\beta\alpha})$$

$$+ \frac{h^2}{40}(z^5-2z^3+z)\,[D_{\alpha\beta\lambda\eta}(f_{\lambda,\eta\beta\alpha}+f_{\eta,\lambda\beta\alpha}) - \tfrac{1}{2}C_{\alpha\beta}p_{,\beta\alpha}]. \quad (38)$$

It is evident that the three dimensional fields $\hat{\mathbf{u}}$, $\hat{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\sigma}}$ are constructed from the two-dimensional variables v_α, w, b_α, $N_{\alpha\beta}$, $M_{\alpha\beta}$ and Q_α of the plate theory introduced in Section 3; in particular, $\hat{\mathbf{u}}$ and $\tilde{\boldsymbol{\sigma}}$ reduce to identities relations (11)–(14), (20) and (21) defining those variables. The fields $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\sigma}}$ are interrelated through the constitutive equations, eqns (6)–(8), and are, therefore, kinematically admissible. The field $\tilde{\boldsymbol{\sigma}}$ satisfies traction boundary conditions (1)–(3) at the faces and with eqns (15) and (16) is seen to meet the equilibrium equations, eqns (4) and (5), thus being statically admissible. Finally, $\hat{\boldsymbol{\sigma}}$ is very close to $\tilde{\boldsymbol{\sigma}}$ but this will be shown later.

The individual components of $\hat{\mathbf{u}}$, $\hat{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\sigma}}$ are derived along with the two-dimensional plate theory equations of Section 3 as follows. First, a guess is made as to the form of the displacement $\hat{\mathbf{u}}$ distribution, guided by previous work on the subject of refined plate theories, notably the one due to Rehfield and Valisetty (1984). The assumed field in eqns (25) and (26) accordingly incorporates the effects of both bending and stretching, yields non-zero transverse shear and normal strains, produces non-linearly distributed surface parallel strains, and by eqns (27) and (28) gives via the constitutive equation, eqn (8), the normal stress $\hat{\sigma}_{33}$ in eqn (32), this well-known distribution being in agreement with traction boundary conditions (2) and (3) on the faces. Also, the form of $\hat{\mathbf{u}}$ conforms to definitions (11), (12) and (20) of w, b_α and v_α which quantities are chosen to be the basic kinematic variables of two-dimensional plate theory.

Introducing eqns (25), (26) and (32) into the constitutive equation, eqn (6), gives $\hat{\sigma}_{\alpha\beta}$ in eqn (30). Since, in general, $\tilde{\boldsymbol{\sigma}}$ is required to be close to $\hat{\boldsymbol{\sigma}}$, one takes $\tilde{\sigma}_{\alpha\beta} = \hat{\sigma}_{\alpha\beta}$ so that eqn (36) holds. Now substituting $\tilde{\sigma}_{\alpha\beta}$ from eqn (30) into eqns (13) and (21) yields after integration over the thickness the constitutive equations, eqns (17) and (23), for the moments $M_{\alpha\beta}$ and forces $N_{\alpha\beta}$. These expressions, in turn, make it possible to rewrite $\tilde{\sigma}_{\alpha\beta}$ as in eqn (33).

Having eqn (33), the equilibrium equation, eqn (4), may be integrated with respect to x_3 in conjunction with traction boundary conditions (1) to find a distribution of $\tilde{\sigma}_{\alpha 3}$. This meets conditions (1) when the in-plane equations of equilibrium (22) are imposed.

Introducing $\tilde{\sigma}_{\alpha 3}$ into eqn (14) gives, in addition, the moment equilibrium equation, eqn (15), with which $\tilde{\sigma}_{\alpha 3}$ finally assumes the form of eqn (34).

Inserting eqn (34) into eqn (5), one may integrate this latter equilibrium equation with respect to x_3 subject to boundary conditions (2) and (3) at the faces in order to get a distribution of $\tilde{\sigma}_{33}$. This distribution will meet boundary condition (3) provided that the transverse force equilibrium equation, eqn (16), holds, and with eqn (16) $\tilde{\sigma}_{33}$ may be expressed as in eqn (35).

Lastly, in order to find $\hat{\sigma}_{\alpha 3}$, expressions (25) and (26) are substituted into the constitutive equation, eqn (7). Since $\hat{\sigma}_{\alpha 3}$ ought to be close to $\tilde{\sigma}_{\alpha 3}$ in eqn (34), relations (29) are adopted for k_α and f_α along with the constitutive equation, eqn (18), for the shear forces Q_α, so that $\hat{\sigma}_{\alpha 3}$ in eqn (31) results, and the derivation is complete.

Preparatory to error analysis, it is expedient to have $\hat{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}$ expressed through the displacements v_α and rotations b_α only. Using eqn (18) and the relation

$$4B_{\alpha 3\eta 3}A_{\alpha 3\lambda 3} = \delta_{\eta\lambda} \tag{39}$$

where $A_{\alpha 3\lambda 3}$ is the inverse of $B_{\alpha 3\eta 3}$ and $\delta_{\eta\lambda}$ stands for the Kronecker delta, f_α in eqn $(29)_2$ can be brought into the form

$$f_\alpha = -\frac{h}{6}g_{,\alpha}-A_{\eta 3\alpha 3}Q_\eta. \tag{40}$$

From eqns (15)–(17) it follows that

$$Q_\alpha = \tfrac{1}{3}h^3D_{\alpha\beta\lambda\eta}(b_{\lambda,\eta\beta}+b_{\eta,\lambda\beta})+O(h^5) \tag{41}$$

$$p = -\tfrac{1}{3}h^3D_{\alpha\beta\lambda\eta}(b_{\lambda,\eta\beta\alpha}+b_{\eta,\lambda\beta\alpha})+O(h^5). \tag{42}$$

Now from eqns $(27)_2$ and $(28)_2$ and (40)–(42) those variables entering $\hat{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}$ that are expressible through rotations b_α have the following estimates:

$$f_\alpha = O(h^3),\quad r = O(h^4),\quad p = O(h^3),\quad Q_\alpha = O(h^3) \tag{43}$$

wherein for the present purpose only h-dependence has been explicitly exposed. Similarly, these contributions to $\hat{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}$ which are functions of the in-plane displacements v_α, by eqns $(27)_1$, $(28)_1$ and $(29)_1$ may be estimated as

$$k_\alpha = O(h^2),\quad l = O(h^3). \tag{44}$$

Use of eqns (43) and (44) in eqns (30)–(32) and (36)–(38) leads to the conclusion that

$$\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}} = O(h^3\hat{\boldsymbol{\sigma}}) \tag{45}$$

where $\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ are understood to be represented by their largest components $\tilde{\sigma}_{\alpha 3}-\hat{\sigma}_{\alpha 3}$ and $\hat{\sigma}_{\alpha\beta}$, respectively, and expressed in terms of v_α and b_α.

5. MEAN SQUARE ERROR ESTIMATE

The stresses $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ can be regarded as elements of function space endowed with the norm

$$\|\tilde{\boldsymbol{\sigma}}\|^2 = \int_F\int_{-h}^{h}[A_{\alpha\beta\lambda\eta}\tilde{\sigma}_{\alpha\beta}\tilde{\sigma}_{\lambda\eta}+4A_{\alpha 3\beta 3}\tilde{\sigma}_{\alpha 3}\tilde{\sigma}_{\beta 3}+2A_{\alpha\beta 33}\tilde{\sigma}_{\alpha\beta}\tilde{\sigma}_{33}+A_{3333}\tilde{\sigma}_{33}\tilde{\sigma}_{33}]\,\mathrm{d}F\,\mathrm{d}x_3 \tag{46}$$

where F is the midplane region and A denotes elements inverse to the components of the elasticity tensor **B**. Obviously, the above energy functional is assumed positive definite to serve as a norm.

If $\boldsymbol{\sigma}$, $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ represent the exact, statically and kinematically admissible stress fields of a given problem in elasticity, the hypersphere theorem due to Prager and Synge (1947) asserts that

$$\|\boldsymbol{\sigma}-\tfrac{1}{2}(\tilde{\boldsymbol{\sigma}}+\hat{\boldsymbol{\sigma}})\|/\|\hat{\boldsymbol{\sigma}}\| = e \tag{47}$$

where

$$e = \tfrac{1}{2}\|\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}\|/\|\hat{\boldsymbol{\sigma}}\|. \tag{48}$$

This implies that $\boldsymbol{\sigma}$ may be approximated by $(\tilde{\boldsymbol{\sigma}}+\hat{\boldsymbol{\sigma}})/2$, the corresponding relative mean square error e being computable from eqn (48). The error decreases with $\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}$ and it was for this reason that one aimed to minimize $\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}$.

The local estimate, eqn (45), will preserve its character in terms of h when the stresses $\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ are replaced by their norms calculated from eqn (46). Thus it follows from eqns (45) and (48) that

$$e = O(h^3/L^3) \tag{49}$$

where L is a characteristic mean square wavelength of the midplane deformation pattern, introduced to ensure that e is a dimensionless quantity. Practically L is calculable knowing the in-plane displacements v_x and rotations b_x from the two-dimensional plate theory. For brevity, one dispenses with recording L explicitly.

The error estimate in eqn (49) signifies that the two-dimensional Reissner theory combined with plane stress theory is capable of providing the three-dimensional stress field $(\tilde{\boldsymbol{\sigma}}+\hat{\boldsymbol{\sigma}})/2$ which differs from the exact elasticity solution $\boldsymbol{\sigma}$ by terms of relative $O(h^3)$. This improves Nordgren's (1972) estimate involving an error of $O(h^2)$ and generalizes Berdichevskii's (1973) h^3 estimate restricted to plates with load-free faces.

6. EFFECT OF EDGE CONDITIONS

Like its predecessors the improved error estimate, eqn (49), is not universally valid for arbitrarily prescribed edge conditions on the cylindrical bounding surface, simply because one constructed $\tilde{\boldsymbol{\sigma}}$ and $\hat{\mathbf{u}}$ paying no attention to those conditions. The artificial edge data conforming to fields $\tilde{\boldsymbol{\sigma}}$ and $\hat{\mathbf{u}}$ in eqns (25), (26) and (33)–(35) are termed "regular", see Koiter (1970). Practically, one is sure to be confronted with irregular situations on the edge and their effect must be taken into account.

Suppose that the edge tractions are irregular, so that the sum $\tilde{\boldsymbol{\sigma}}+\boldsymbol{\sigma}^*$ must be used instead of $\tilde{\boldsymbol{\sigma}}$ alone to satisfy the true conditions, $\boldsymbol{\sigma}^*$ being the corresponding edge-zone correction. Replacing in eqn (48) $\tilde{\boldsymbol{\sigma}}$ by $\tilde{\boldsymbol{\sigma}}+\boldsymbol{\sigma}^*$ and using the triangle inequality, one finds that the error is now bounded by $e+e^*$, e having its previous meaning, eqn (48), and estimate (49), and e^* denoting a supplement of the form $2e^* = \|\boldsymbol{\sigma}^*\|/\|\hat{\boldsymbol{\sigma}}\|$. As Koiter (1970) points out, $\hat{\boldsymbol{\sigma}}$ extends over a distance of $O(L)$ from the edge, while $\boldsymbol{\sigma}^*$ may be restricted to a narrow zone of width h near the boundary. Accordingly, from eqn (46), e^* can be evaluated

$$e^* = O(h/L)^{1/2}|\boldsymbol{\sigma}^*|/|\hat{\boldsymbol{\sigma}}| \tag{50}$$

where $|\boldsymbol{\sigma}^*|$ and $|\hat{\boldsymbol{\sigma}}|$ are the absolute maximum values of $\boldsymbol{\sigma}^*$ and $\hat{\boldsymbol{\sigma}}$. Clearly, the order of the term $|\boldsymbol{\sigma}^*|/|\hat{\boldsymbol{\sigma}}|$ will depend on the concrete form of the boundary conditions. Consider, as an example, the free edge. The exact requirement that $\boldsymbol{\sigma}\cdot\mathbf{n} = 0$, $\mathbf{n}$ being the unit normal to the edge, is violated by the interior stress field $\tilde{\boldsymbol{\sigma}}$ in eqns (33) and (34) because of the k_x, f_x and p terms. In view of eqns (43) and (44), the edge correction $\boldsymbol{\sigma}^*$ corresponding to those terms will be of $O(h^2)$ relative to $\hat{\sigma}$ and the error e^* from irregular edge data will become by eqn (50) proportional to $h^{5/2}$, a slightly greater value than that in eqn (49) for regular conditions.

It should be emphasized that our estimates of e and e^* are global (based on integral norm (46)) and as such do not guarantee local closeness of our statically and kinematically admissible solutions to the exact three-dimensional elasticity solutions, especially near the edges of the plate. The largest local error is to be expected when the exact solution involves stress singularities, as happens in plates with clamped edges.

7. CONCLUDING REMARKS

This paper proves that Reissner's bending theory combined with the plane stress theory is capable of predicting the three-dimensional behavior of homogeneous elastic plates within a relative mean square error of $O(h^3)$ compared with elasticity solutions for arbitrarily distributed surface lateral loading, provided that there are regular boundary conditions on the edge. In contrast to known results in the literature, this refined error estimate corresponds to Reissner's theory in its general form which incorporates the transverse normal stress effect in constitutive equations.

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